ON THE DEFINITION OF THE FATIGUE LOADING FOR SAND

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Abstract: High-cycle models may be used to estimate the cumulative effects in soil due to many (millions) of cycles of relatively small strain amplitude (10^{-4}). Such models have been described in the literature and they may be useful to estimate settlements or for designing compaction strategies. This paper demonstrates some problems related to the description of fatigue loading. A single fatigue load increment is a package of cycles quantified by a periodic strain path and by the number of cycles. Some improvements in the definition of the amplitude are proposed, in particular how to detrend the strain path and how to isolate the individual oscillations from a complex 6-dimensional signal. We also present an experimental contribution to the discussion on the Miner's rule for cycles showing different polarizations.

INTRODUCTION

Our high-cycle model [2, 4] takes a package of strain cycles as a single input increment and returns the accumulated strain or stress. Such packages of cycles are single increments of fatigue loading. Our high-cycle accumulation model deals with the cumulative deformation due to a large number (say 10^3 to 10^7) of load cycles at a small strain amplitude (less than 10^{-3}).

The semi-empirical equations in [2] based on experimental evidence are relatively simple so we do not repeat them here for the sake of space. A version of the manuscript of [2] can be downloaded from our homepage [2]. We strongly recommend to have [2] at hand while reading this paper.

The fatigue has a different meaning for sand than the usual susceptibility to failure due to cyclic loading (= CL). CL causes merely an accumulation of deformation or stress relaxation in soil. Contrarily to the fatigue in metals the strength of soil usually increases during CL due to densification. The only exception is the 'undrained strength' which can temporarily decrease due to CL under undrained conditions because the pore pressure build-up leads to a decrease of the effective pressure.

In the simplest case the fatigue loading can be regarded as the number of cycles N_c weighted by the square of the strain amplitude ϵ^{ampl}, which can be written as $\int(\epsilon^{\text{ampl}})^2 \mathrm{d}N_c$. The deformation paths that result from moving traffic loads or from compaction machines or generally from dynamic excitation may be quite complex, however. Some mathematical tools are needed to handle them (e.g. rainflow counting) and to estimate their amplitudes, ovalities and polarizations. Our definition [2] of amplitude may lead to inaccurate predictions of the accumulation rate, e.g. in cases of strain paths of somewhat artificial shapes like cross, or diamond. Despite identical size, two seemingly similar cross-like loops, Fig.1-b and Fig.1-d, lead to a different accumulation.

Yet another problem is related to the counting of cycles. A computational splitting of a single X-cross loop, Fig. 1-b, into a sequence of *two* cycles with a rapid polarization change of $90°$ after each cycle will significantly overestimate the accumulation (because of the large value of f_π as defined in [2]).

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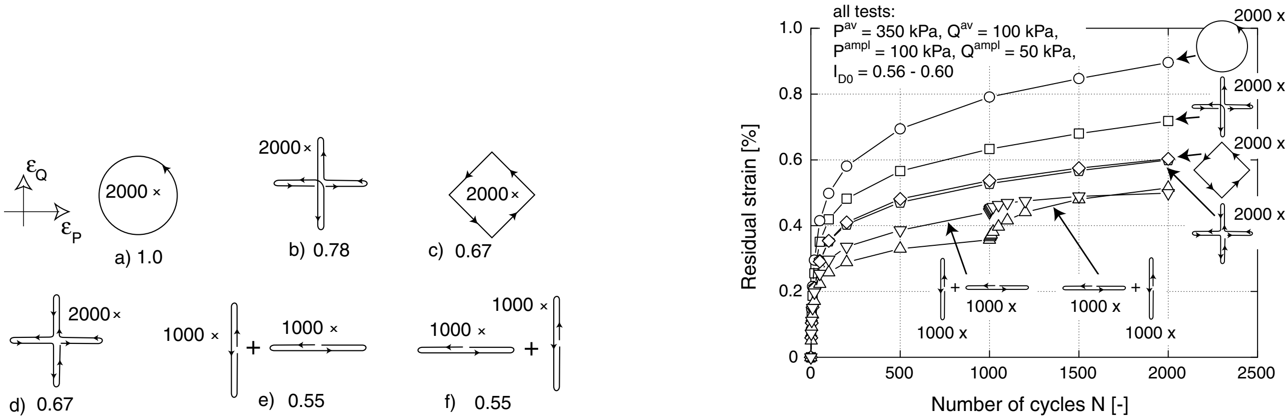


Figure 1: *Left: The shape of cycles (despite of equal spans) affect the rate of accumulation. The accumulation caused by the circular cycles is taken as the reference (=100%). This effect has not been included in the model [2] as yet. Right: the accumulation measured in the laboratory*

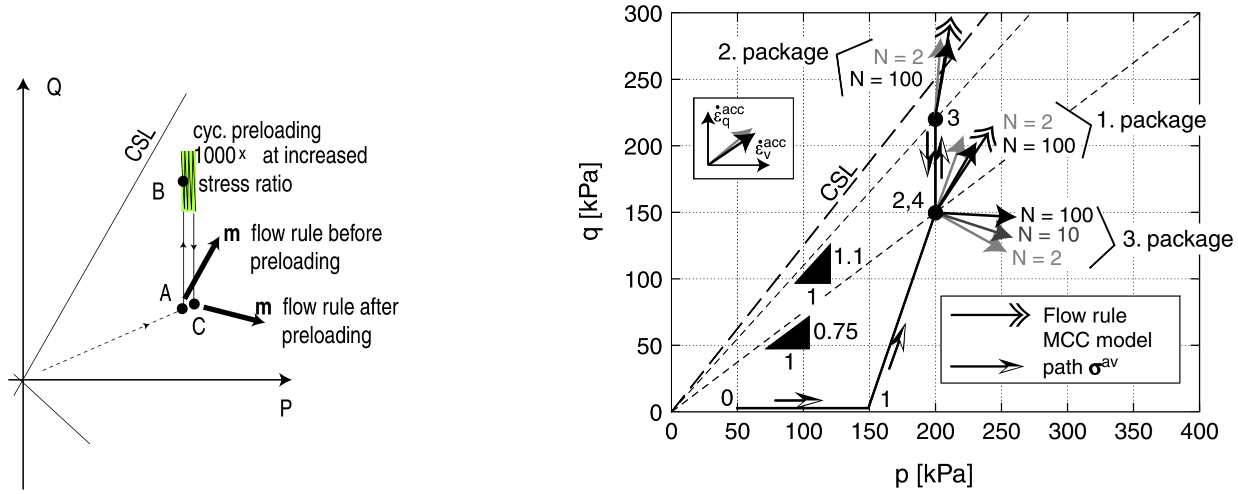


Figure 2: *Cyclic preloading at an increased stress ratio affects the rate and the direction of accumulation. This effect has not been included in the model [2, 3] as yet. Left: the idea of the test; Right: test results*

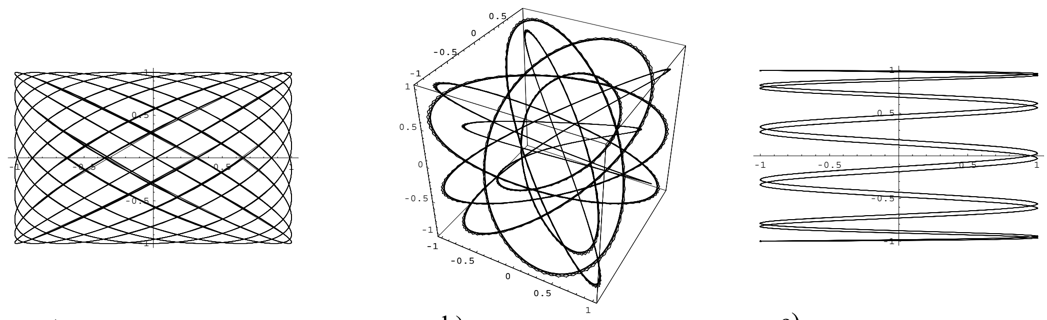
The laboratory tests, Fig. 1, give a deeper insight into the validity of the Palgram-Miner's rule [1] for sand. In [2] we have demonstrated that the sequence of application of packages with different amplitudes does not influence the total accumulation effect. A pleasant observation from the recent tests is that the same holds for packages with different polarizations, cf. Fig.1-e and -f.

The high-cycle model can predict fairly well the accumulation due to a large number of strain cycles that follows a monotonic loading. However, if such cyclic loading is preceded by a *cyclic preloading* at considerably different average stress, Fig. 2, then both the cyclic flow rule and the expression for the intensity of accumulation proposed in the model become inaccurate.

TOWARDS A BETTER DEFINITION OF THE FATIGUE LOADING

In [2] we have already given arguments for expressing both the amplitude and the accumulation in terms of strain rather than stress. We have also discussed how to obtain the strain amplitude from stress- or mixed - controlled tests. However, several questions related to the definition of the amplitude still remain open:

- After a full cycle the strain path does not exactly pass through the same strain state (due to accumulation). Moreover the strain loop may intersect itself like in Fig. 1-b which does not indicate that the loop is over. It is evident that a mathematical tool is required in order to detect the period, i.e. when a strain loop is finished.
- Suppose a strain loop has been prescribed by two spans with slightly different frequencies so that a slow rotation of polarization occurs, Fig. 3 a,b. If two spans were equally polarized the beat would occur. The hitherto hypothesis either ignores the small spans or overestimates its effect describing such loading as distinct packages with alternating polarization.



a) b) c)

Figure 3: *Strain paths (Lissajous curves) obtained from the superposition of sine functions with slightly different frequencies and amplitudes a) and b) or with strongly different frequencies but similar amplitudes c)*

- It is not clear if smaller but faster cycles in plane with the dominant cycle or out of this plane may be ignored, Fig. 4.

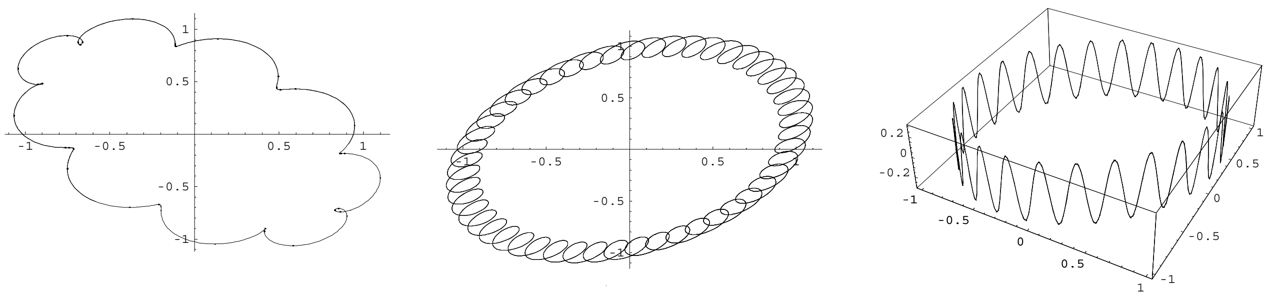


Figure 4: *Strain paths obtained from the summation of sine functions with very different frequencies and different amplitudes.*

DETRENDING OF THE STRAIN PATH

We deal with a strain path $\boldsymbol{\epsilon}(t)$ or with a stress path $\boldsymbol{\sigma}(t)$ assuming that its 6 components $\epsilon_{ij}(t)$ are given (usually we have a list of 6 strain components parametrized with time t).

Note that contrarily to the strain $\boldsymbol{\epsilon}$ or the strain rate $\mathbf{D}$ one cannot find the principal values of the strain *path* $\boldsymbol{\epsilon}(t)$ so all 6 components must be considered.

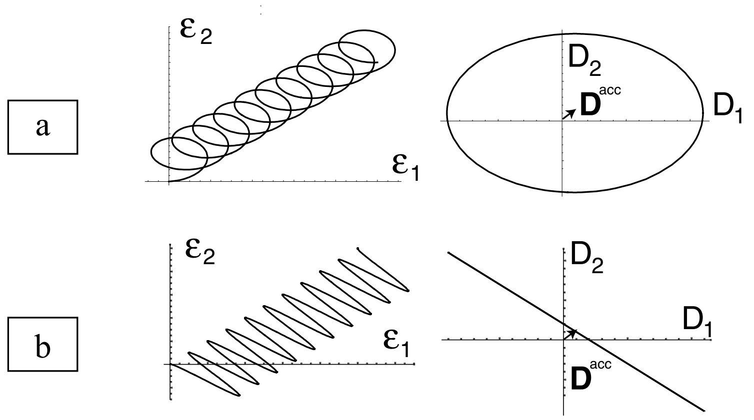


Figure 5: *A hodograph is a trajectory of* $\mathbf{D}(t) \approx \dot{\boldsymbol{\epsilon}}(t)$ *parametrized with time* t*, analogously to the strain path* $\boldsymbol{\epsilon}(t)$*. The rate of accumulation can be easily identified as a drift rate (denoted with arrow) of the average strain upon a cycle. Note that the strain rate is an exactly periodic function* $\mathbf{D}(t) = \mathbf{D}(t + NT)$ *whereas the strain* $\boldsymbol{\epsilon}(t)$ *is not. The distinction between the cycles encompassing some area (out-of-phase cycles (= OOP) ,* $\boxed{\text{a}}$*) and the open-curve cycles (in-phase cycles (= IP),* $\boxed{\text{b}}$ *) is of importance.*

The resilient strain path $\epsilon^e_{ij}(t)$ is obtained from the original (recorded or calculated) signal $\epsilon_{ij}(t)$ by subtracting the residual (cumulative) portion (pseudo-creep) from it. This process is known as a *detrending*.

The proposed *detrending procedure* consists of three steps:

- calculate a hodograph $\dot{\epsilon}_{ij}(t)$, Fig. 5
- find the average point of the hodograph $\dot{\epsilon}^{\text{av}}_{ij}$
- subtract the cumulative portion from the original path: $\epsilon^e_{ij}(t) = \epsilon_{ij}(t) - \dot{\epsilon}^{\text{av}}_{ij} t$

Analogously, we may proceed with the stress path removing the cumulative portion (pseudo-relaxation): $\sigma^e_{ij}(t) = \sigma_{ij}(t) - \dot{\sigma}^{\text{av}}_{ij} t$. Possibly we are given a mixed data with some stress components σ_{ij} and some strain components $\epsilon_{i'j'}$ with complementary components i', j' with respect to i, j. In such mixed case the removal of the residual part can be performed componentwise. Next, the stress components are converted to the strain components using elasticity (elastic material parameters E, ν assumed as given), i.e. the system of equations $\sigma^e_{ij} = E_{ijkl} \epsilon^e_{kl}$ is solved for the missing components of ϵ^e_{kl}.

SPECTRAL ANALYSIS

The detrended strain path is assumed to be a superposition of individual harmonic signals in the direction of each strain component. The harmonic signals can be distinguished judging by the frequency f_K (or angular velocity $\omega_K = 2\pi f_K$). From each of six components $\epsilon_{ij}(t)$ of the strain path we pick up a portion which corresponds to a common dominant frequency f_K. We put these six signals together and call this sum *oscillation*. In general it is a 6-dimensional ellipse in the strain space. In this text the oscillations are numbered with the capital letter K.

We will try to approximate the signal $\epsilon_{ij}(t)$ as a sum of M oscillations:

$$\epsilon_{ij}(t) \approx \sum_{K=1}^{M} \epsilon_{ij}^{\mathrm{ampl}K} \sin(\omega^K t + \varphi_{ij}^K) \tag{1}$$

In the detrended strain portion $\epsilon_{ij}^{e}(t)$ we have omitted the index ^{e} for brevity.

Apart from the angular velocity ω^K the other parameters of an oscillation are the amplitude $\epsilon_{ij}^{\mathrm{ampl}K}$ and the phase shift φ_{ij}^K. They are assumed to remain nearly constant so that the size and the ovality of the oscillation do not change.

The essential purpose of the present spectral analysis is filtering out spectral components corresponding to the same angular velocity ω^K from all strain components ϵ_{ij} and collecting them to a common oscillation. For simplicity this is done only for several dominant frequencies f^K with $K = 1, 2, \dots$ for which the strain amplitudes $\epsilon_{ij}^{\mathrm{ampl}\ K}$ are large. Since the square of the amplitude dictates the accumulation rate the impact of small amplitudes is almost negligible. For example we may easily neglect amplitudes say 10-times smaller than the largest one at the cost of 1% error only (this is a rough estimation which assumes that the respective frequencies are not very different).

Each component function $\epsilon_{ij}(t)$ is treated as a series of discrete values $\epsilon_{ij(k)}$ given at $k = 0, 1, \dots, N-1$ (N is an even number) points equally distributed along the time axis over the time window from $t = 0$ to $t = (N-1)\Delta$.

Denoting the sampling interval as Δ we find the Nyquist frequency $f_c = 1/(2\Delta)$ and the (complex valued) discrete Fourier transform (DFT) $Y_{ij(n)}$ of the discrete strain path $\epsilon_{ij(k)}$ with frequencies $f_n = \omega_n/(2\pi)$ such that $-f_c < f_n < f_c$. The frequencies are indexed with $n = -N/2, \dots, -1, 0, 1, \dots N/2$ and

$$Y_{ij(n)} \equiv \sum_{k=0}^{N-1} \epsilon_{ij(k)} e^{2\pi I k n/N} \tag{2}$$

where $I^2 = -1$. This DFT approximates the Fourier transform $Y_{ij}(f)$ at discrete frequencies f_n according to $Y_{ij}(f_n) \approx Y_{ij(n)}\Delta$ but we will use the DFT only. We assume that our sampling interval Δ is sufficiently small to capture all strain oscillations of importance and that no higher frequencies can leak into the $(-f_c, f_c)$ range (no aliasing).

Plotting the tensorial norm (Einstein summation over the tensorial components $_{ij}$) we obtain the periodogram

$$Y_{(n)} = \begin{cases} |Y|_{ij(n)}|Y|_{ij(n)} + |Y|_{ij(-n)}|Y|_{ij(-n)} & \text{for} \quad n = 1, 2 \dots, N/2-1 \\ |Y|_{ij(0)}|Y|_{ij(0)} & \text{for} \quad n = 0 \\ |Y|_{ij(N/2)}|Y|_{ij(N/2)} & \text{for} \quad n = N/2 \end{cases} \tag{3}$$

and among the frequencies $f_n = n/(N\Delta)$ we may find the one for which the (real valued) function $Y_{(n)}$ has its maximum. This frequency is denoted as f_K and the corresponding angular velocity $\omega_K = 2\pi f_K$ enters (1). Technically, since the original signal $\epsilon_{ij}(t)$ may be a pure sine function with a frequency lying in the middle *between* two adjacent f_n-s one needs data windowing (apodization, e.g. Hann or Barlett window) in order to reduce the leakage of frequency.

Having found the dominant frequency f_K we filter out a band around this frequency from each component of the strain. For this purpose we simply multiply each DFT $Y_{ij}(f_n)$ by the band-pass filter H_{BP}^K (an even function equal to unity in the vicinity of $\pm f_K$ and to zero elsewhere) in the frequency domain. We obtain six f_K-band-pass filtered transforms

$$Y_{ij(n)}^K = H_{BP}^K Y_{ij(n)} \tag{4}$$

which constitute the DFT of the K-th oscillation. The amplitudes $\epsilon_{ij}^{\text{ampl } K}$ of the strain components are obtained from the discrete inverse Fourier transform (DIFT)

$$\epsilon_{ij(k)} = \frac{1}{N}\sum_{n=0}^{N-1} Y_{ij(n)}^K e^{-2\pi I kn/N} \tag{5}$$

Among all k-indexed values we find the difference between maximum of $\epsilon_{ij(k)}$ and minimum of $\epsilon_{ij(k)}$ (for each component $_{ij}$ separately). These differences correspond to the double amplitudes $2\epsilon_{ij}^{\text{ampl}K}$ of the oscillation K and enter (1). The expression for DIFT contains an n-sum from 0 to N (instead of from $-N/2$ to $N/2$) thanks to the N-periodicity of the DFT, i.e. $Y_{ij(-n)} = Y_{ij(N-n)}$.

The phase shift φ_{ij}^K is calculated from the correlation of individual components $_{ij}$. For example, we may assume $\varphi_{11}^K = 0$ and the phase shift φ_{22}^K is calculated in the frequency domain using the product $Y_{11}^K\ Y_{22}^{*K}$ wherein the asterisk denoted the complex conjugation. The phase shift follows from the time lag τ_{22} obtained as the time shift for which the DIFT of the above product has the maximum. Finally we have $\varphi_{22}^K = \omega^K \tau_{22}$. The K-th oscillation is completely determined by repeating analogous calculations of correlation for all strain components $_{ij}$.

The remaining oscillations are selected analogously using the *reduced* signal

$$Y_{ij(n)} = H_{NO}^K Y_{ij(n)}, \tag{6}$$

where H_{NO}^K denotes the notch filter (an even function equal to zero in the vicinity of $\pm f_K$ and to unity elsewhere). Currently, the fatigue load contributions from the individual oscillations i.e. the size of the amplitude $\epsilon^{\text{ampl}K} = \|\boldsymbol{\epsilon}^{\text{ampl}K}\|$ and the number of cycles N_c enter the fatigue loading independently. A single load package from a time period T is calculated as $T\sum_K f_K \left[\epsilon^{\text{ampl}K}\right]^2$ i.e. without considering the mutual polarizations of different oscillations within the package. One of the difficulties is the lack of knowledge about accumulation under such complex loading conditions. If we split the oscillations treating them as short separate packages repeatedly applied *after each other* then the factor f_π, responsible for a change of polarization in the model [2], will lead to a significant overestimation of the accumulation rate.

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